

A Gorenstein numerical semi-group ring having a transcendental series of Betti numbers.

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0. Introduction

Around 1969 Oscar Zariski asked Ernst Kunz whether there were any relations between Gorenstein rings and symmetric numerical semigroup rings (cf. the introduction to [Kunz], where this question was settled by Kunz ; more details below). Presumably Zariski was inspired both by the thesis of Jürgen Herzog [Her] and by his own results that to *plane curve singularities* there is always associated a numerical semigroup ring which was a *complete intersection* (i.e. a special case of a Gorenstein ring) [Zar]. According to Tate [Ta] a complete intersection always has a rational Poincaré-Betti series. Here we will prove that there are Gorenstein numerical semigroup rings whose Poincaré-Betti series is transcendental. More precisely, here is the smallest example we have been able to find: let k be a field of characteristic zero and R the subring of the polynomial ring $k[t]$, generated by the twelve monomials

$$t^{36}, t^{48}, t^{50}, t^{52}, t^{56}, t^{60}, t^{66}, t^{67}, t^{107}, t^{121}, t^{129}, t^{135}$$

Then R is a Gorenstein ring and the generating series (the Poincaré-Betti series of R)

$$P_R(z) = \sum_{i \geq 0} |\mathrm{Tor}_i^R(k, k)| z^i$$

is rationally related to the infinite product

$$\prod_{n=1}^{\infty} \frac{(1 + z^{2n-1})^2}{(1 - z^{2n})^2}$$

and thus transcendental. If we divide R by the non-zero divisor in the maximal ideal corresponding to e.g. t^{36} we obtain an *artinian* Gorenstein ring of embedding dimension 11 having an irrational series of Betti numbers and this might be the smallest possible example, cf. Bøgvad [Bø] and [Roos1, pages 459–461]. Cf. also [Roos2] for the skew-commutative case. For even more precise results and details, cf. Theorem 1 below. But even more interesting are the methods of proofs related to Golod homomorphisms, so-called large maps, graded Lie algebras etc. and thereby to results by Gulliksen, Levin, Avramov, Löfwall, Bøgvad and others.

1. Symmetric numerical semigroups and Gorenstein rings

A numerical semigroup S is an additive semigroup of natural numbers generated by a set of numbers $0 < g_1 < g_2 < \dots < g_n$ such that $\gcd(g_1, g_2, \dots, g_n) = 1$. We write $S = (g_1, \dots, g_n)$. Let k be a field of characteristic zero. The numerical semigroup ring $k[S]$ of S over k is by definition the subring of the polynomial ring $k[t]$ generated by the monomials

$$t^{g_1}, t^{g_2}, \dots, t^{g_n}$$

The Poincaré-Betti series of $k[S]$ is by definition the generating series

$$P_{k[S]}(z) = \sum_{i \geq 0} |\mathrm{Tor}_i^{k[S]}(k, k)| z^i$$

(for a k -vector space V we denote by $|V|$ the dimension of V over k). In [Fro-Ro] Ralf Fröberg and myself found by modifying an earlier result by Roos-Sturmfels [Ro-St] that the following semigroup

$$S = (18, 24, 25, 26, 28, 30, 33)$$

and the corresponding $k[S]$ had a series $P_{k[S]}(z)$ that was an explicit transcendental function. In the present paper I will prove that there are semigroups S such that $G = k[S]$ is a *Gorenstein* ring and such that $P_G(z)$ is a transcendental function.

Let us recall that $\mathbf{N} \setminus S$ is finite, and let $F(S)$ the largest integer that is not in S (it is called the Frobenius number of S). Now S is called symmetric if for every n either $n \in S$ or $F(S) - n \in S$. The following result was proved in the paper [Kunz] (mentioned in the introduction): $k[S]$ is a Gorenstein ring if and only if S is symmetric. Now in [Ro-G-San] there are described infinitely many different ways to associate a symmetric semigroup to a given semigroup and we will see that from *a homological point of view* these different ways give essentially the same result. Let us start with any numerical semigroup S and let us recall the recipe from [Ro-G-San]. Let $F(S)$ be the Frobenius number of S just defined ($F(S) + 1$ is also called the conductor of S). Let us also denote by $PF(S)$ the set of pseudo-Frobenius numbers of S , i.e. those integers z such that $z + S \setminus \{0\} \subseteq S$. The cardinality of $PF(S)$ is also called the type of S . Now suppose that $S = (g_1, \dots, g_n)$ and that $PF(S) = (n_1, \dots, n_t)$ and let \bar{g} be an *odd* integer such that $\bar{g} \geq 3F(S) + 1$. Then

$$\bar{S}_{\bar{g}} = (2g_1, 2g_2, \dots, 2g_n, \bar{g} - 2n_1, \bar{g} - 2n_2, \dots, \bar{g} - 2n_t)$$

is a symmetric numerical semigroup such that $S = \{n | 2n \in \bar{S}_{\bar{g}}\}$. Note that this gives infinitely many $\bar{S}_{\bar{g}}$ but they are all *essentially* the same from a homological point of view. We will only illustrate this with the special case $S = (18, 24, 25, 26, 28, 30, 33)$ taken from [Fro-Ro] and [Ro-St]. Now in this case $F(S) = 65$ and $PF(S) = (65, 45, 38, 34, 31)$. Thus the odd integers $\geq 3F(S) + 1$ are the integers 197, 199, 201, \dots and

$$\bar{S}_{197} = (36, 48, 50, 52, 56, 60, 66, 67, 107, 121, 129, 135)$$

and

$$\bar{S}_{199} = (36, 48, 50, 52, 56, 60, 66, 69, 109, 123, 131, 137)$$

etc. We start by analyzing the smallest case \bar{S}_{197} . Let us denote the corresponding numerical semigroup ring by R_{197} which is a Gorenstein ring of dimension 1 and a domain, since it is a subring of $k[t]$.

We therefore have to determine the series

$$(2) \quad P_{R_{197}}(x, y) = \sum_{i,j} |\mathrm{Tor}_{i,j}^{R_{197}}(k, k)| x^i y^j$$

where j refers to the grading of R_{197} . For this analysis we use Macaulay2 [M2] (working over \mathbf{Q}). Recall that the ring R_{197} can be obtained by introducing a ring $R = \mathbf{Q}[t]$, another ring $T = \mathbf{Q}[a, b, c, d, e, f, g, h, i, j, k, l]$ and a map ϕ between them which is defined by $a \rightarrow t^{36}, b \rightarrow t^{48}$ etc. The kernel of ϕ is an ideal J in T and

$$\mathbf{Q}[t^{36}, t^{48}, t^{50}, t^{52}, t^{56}, t^{60}, t^{66}, t^{67}, t^{107}, t^{121}, t^{129}, t^{135}] \cong T/J \cong R_{197}$$

It is important for us to keep track of the gradings and therefore the Macaulay2 code is:

```
R:=QQ[t]
T:=QQ[a..l, Degrees=> {{36},{48},{50},{52},{56},{60},{66},{67},{107},{121},{129},{135}}]
phi=map(R,T,{t^36,t^48,t^50,t^52,t^56,t^60,t^66,t^67,t^107,t^121,t^129,t^135})
ker phi
J=trim(oo)
```

The ideal J obtained is minimally generated by the following elements:

$$b^2 - af, c^2 - bd, cd - ag, d^2 - be, de - bf, a^3 - bf, e^2 - df, ef - cg, a^2b - f^2, a^2c - eg, a^2f - g^2, abc - h^2, adh - bi,$$

$$\begin{aligned}
& ci - aj, aeh - di, afh - ei, bch - ak, bdh - fi, agh - bj, cj - al, beh - al, gi - dj, ceh - dj, bfg - hi, ej - bk, \\
& cfh - bk, a^2i - ck, dfh - ck, fj - dk, bgh - dk, cgh - bl, ek - cl, dgh - cl, gj - dl, f^2h - dl, eg^2 - hj, egh - fk, abi - el, \\
& fgh - a^2j, gk - fl, adi - fl, abdf - hk, a^2k - gl, abdg - hl, adfg - i^2, afg^2 - ij, bhj - ik, j^2 - il, bfh^2 - il,
\end{aligned}$$

$$(3) \quad jk - bhl, fhk - jl, k^2 - ehl, aij - kl, ei^2 - l^2$$

As we said above the ring $R_{197} = T/J$ is a Gorenstein ring of Krull dimension 1 and it is a subring of $\mathbf{Q}[t]$, thus a domain. We now divide out by the non-zero divisor a of degree 36. Now $(T/J)/(a)$ is an artinian Gorenstein ring which also can be quickly determined by Macaulay2 as follows: We start by simplifying the ideal J by putting $a = 0$

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substitute(J, {a=>0})
I=trim(oo)

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We get a new graded ideal I which is easier, since it is minimally generated by:

$$\begin{aligned}
& b^2, c^2 - bd, cd, d^2 - be, bf, de, e^2 - df, ef - cg, f^2, eg, g^2, h^2, bi, ci, di, ei, bch, bdh - fi, bj, cj, beh, gi - dj, \\
& ceh - dj, hi, ej - bk, cfh - bk, ck, dfh, fj - dk, bgh - dk, cgh - bl, ek - cl, dgh - cl, dl, gj, hj, fk, el, \\
& fgh, fl, gk, hk, gl, hl, i^2, ij, ik, il, j^2, jk, jl, k^2, kl, i^2
\end{aligned}$$

and it can be considered in the ring $\mathbf{Q}[b, c, d, e, f, g, h, i, j, k, l]$, so that $R_{197}/(a) \cong \mathbf{Q}[b, c, d, e, f, g, h, i, j, k, l]/I$ and furthermore

$$(4) \quad P_{R_{197}/(a)}(z, 1) = P_{R_{197}}(z, 1)/(1 + z)$$

since a is a nonzero divisor. Note that in (4) we have taken the total degree, but everything is still also graded as follows:

$$b, c, d, e, f, g, h, i, j, k, l \text{ have the degrees } 48, 50, 52, 56, 60, 66, 67, 107, 121, 129, 135$$

but it is unwieldly to work with such high degrees. We therefore check the possible gradings of the last ideal I . We therefore temporarily denote the possible degrees of the variables $b, c, d, e, f, g, h, i, j, k, l$ by the same letters, and in order to find all those integral degrees of the variables for which the relations in I are still homogeneous, we have to solve the following linear equations for integer solutions corresponding the the non-monomial (i.e. the binomial) relations in I , where the relation $c^2 - bd$ gives the linear equation $2c - b - d = 0$ etc.:

$$\begin{array}{lll}
2c - b - d = 0 & g + i - d - j = 0 & b + g + h - d - k = 0 \\
2d - b - e = 0 & c + e + h - d - j = 0 & c + g + h - b - l = 0 \\
2e - d - f = 0 & e + j - b - k = 0 & e + k - c - l = 0 \\
e + f - c - g = 0 & c + f + h - b - k = 0 & d + g + h - c - l = 0 \\
b + d + h - f - i = 0 & f + j - d - k = 0 &
\end{array}$$

We have 14 equations for the 11 unknowns $b, c, d, e, f, g, h, i, j, k, l$ and the result is: (we have three constants c_1, c_2, c_3):

$$\begin{aligned}
(5) \quad & b = c_1 & h = c_3 \\
& c = (c_1 + c_2)/2 & i = c_3 - 2c_2 + 3c_1 \\
& d = c_2 & j = (2c_3 + 3c_2 - c_1)/2 \\
& e = 2c_2 - c_1 & k = (2c_3 + 7c_2 - 5c_1)/2 \\
& f = 3c_2 - 2c_1 & l = c_3 + 5c_2 - 4c_1 \\
& g = (9c_2 - 7c_1)/2 &
\end{aligned}$$

Thus $c_1 = 1$ and $c_2 = 1$ give the solutions $b = c = d = e = f = g = 1$ and $h = c_3$, $i = j = k = l = c_3 + 1$ so that the minimal choice for a positive integral grading is $c_3 = 1$ which gives the final minimal integral grading $b = c = d = e = f = g = h = 1$, $i = j = k = l = 2$. Now we transform now our previous ideal I to a ring with this last grading:

```
JE:=QQ[b..1,Degrees => {{1},{1},{1},{1},{1},{1},{1},{1},{2},{2},{2},{2}}]
I2 = substitute(I,JE)
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Thus we now have a ring $\bar{R}_{197} \cong JE/I2$ which we will study in detail.

The maximal ideal $(b, c, d, e, f, g, h, i, j, k, l)$ is generated by b, c, d, e, f, g, h of degree 1 and i, j, k, l of degree 2. The square of the maximal ideal is generated by $gh, fh, eh, dh, ch, bh, fg, dg, cg, bg, df, cf, ce, be, bd, bc$ and the cube of the maximal ideal is generated by $cl, bl, dk, bk, dj, fi, bdg$ and finally the fourth power of the maximal ideal is generated by bcl which is also the socle of \bar{R}_{197} . This and the relations $I2$ show that $\bar{R}_{197} \cong JE/I2$ is the trivial extension of the ring

$$(6) \quad \mathcal{S} = \mathbf{Q}[b, c, d, e, f, g]/(b^2, c^2 - bd, cd, d^2 - be, de, bf, e^2 - df, ef - cg, eg, f^2, g^2)$$

with an \mathcal{S} -module M that is generated by h, i, j, k, l in $\bar{R}_{197} \cong JE/I2$. Thus $\bar{R}_{197} = \mathcal{S} \ltimes M$ and M can be defined as the cokernel of the map

$$\mathcal{S}^{28} \longrightarrow \mathcal{S}^5$$

defined by the 5×28 matrix over \mathcal{S} : (the generators h, i, j, k, l correspond to the 5 rows of this matrix):

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & fg & dg & cg & bg & df & cf & ce & be & bd & bc \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g & e & d & c & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -f & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g & f & e & c & b & -d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g & f & e & c & 0 & -d & -b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -d & 0 & -b & 0 & 0 & 0 & 0 & 0 \\ g & f & e & d & 0 & 0 & -c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c & -b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Example: left matrix multiplication of the row matrix h, i, j, k, l in JE with the seventh column of the matrix above gives the relation $ke - cl$ in JE .

Next we use the result proved by Avramov and Levin [Av-Le] that since \bar{R}_{197} is Gorenstein the natural map $\bar{R}_{197} \longrightarrow \bar{R}_{197}/(bcl)$ is a Golod map. Let us put $\bar{R} = \bar{R}_{197}/(bcl)$ With this notation the Golod condition implies that

$$P_{\bar{R}}(z) = \frac{P_{\bar{R}_{197}}}{1 - z^2 P_{\bar{R}_{197}}(z)}$$

which we will write in the form

$$(7) \quad \frac{1}{P_{\bar{R}_{197}}(z)} = \frac{1}{P_{\bar{R}}(z)} + z^2$$

Now bcl lies in M and therefore \bar{R} is a new trivial extension $\mathcal{S} \ltimes \bar{M}$ where $\bar{M} = M/(bcl)$. Now, according to a result of Gulliksen [Gu]

$$(8) \quad P_{\bar{R}}(z) = \frac{P_{\mathcal{S}}(z)}{1 - z P_{\bar{M}}(z)}$$

In order to determine $\text{Ext}_{\mathcal{S}}^*(\bar{M}, k)$ we first observe that among the generators of \bar{M} , h plays a special role since it has degree 1, while the other generators i, j, k, l have degree 2. Let N be the submodule of \bar{M} generated by h . We have an exact sequence of \mathcal{S} -modules:

$$(9) \quad 0 \longrightarrow N \longrightarrow \bar{M} \longrightarrow \bar{M}/N \longrightarrow 0$$

which gives a long exact sequence of $\text{Ext}_{\mathcal{S}}^*(k, k)$ -modules:

$$\dots \rightarrow \text{Ext}_{\mathcal{S}}^n(\bar{M}/N, k) \rightarrow \text{Ext}_{\mathcal{S}}^n(\bar{M}, k) \rightarrow \text{Ext}_{\mathcal{S}}^n(N, k) \rightarrow \text{Ext}_{\mathcal{S}}^{n+1}(\bar{M}/N, k) \rightarrow \text{Ext}_{\mathcal{S}}^{n+1}(\bar{M}, k) \rightarrow \dots$$

I now claim that

$$(10) \quad \text{Ext}_{\mathcal{S}}^*(\overline{M}, k) \longrightarrow \text{Ext}_{\mathcal{S}}^*(N, k)$$

is an epimorphism. But (10) is a map of (left) $\text{Ext}_{\mathcal{S}}^*(k, k)$ -modules. The important thing now is that (10) is an epimorphism in degrees $*$ = 0 and $*$ = 1 (direct calculation). Therefore if we prove that $\text{Ext}_{\mathcal{S}}^*(N, k)$ is generated as a (left) $\text{Ext}_{\mathcal{S}}^*(k, k)$ -module by its elements in degree 0 and 1 it will follow that (10) is an epimorphism in all degrees. But the annihilator of (h) in $\overline{R} = \mathcal{S} \propto \overline{M}$ is $(fg, df, be, bc, bdg, h, i, j, k, l)$. Thus we have an exact sequence of \mathcal{S} -modules:

$$(11) \quad 0 \longrightarrow (fg, df, be, bc, bdg) \longrightarrow \mathcal{S} \longrightarrow N \longrightarrow 0$$

But the five generating elements in the ideal to the left in (11) are all annihilated by the maximal ideal in \mathcal{S} . Thus if we take $\text{Ext}_{\mathcal{S}}^*(\cdot, k)$ of the sequence (11) it follows that $\text{Ext}_{\mathcal{S}}^*(N, k)$ is indeed generated as an $\text{Ext}_{\mathcal{S}}^*(k, k)$ -module by its elements of degree 0 and 1. Thus we have indeed a short exact sequence

$$(12) \quad 0 \longrightarrow \text{Ext}_{\mathcal{S}}^*(\overline{M}/N, k) \longrightarrow \text{Ext}_{\mathcal{S}}^*(\overline{M}, k) \longrightarrow \text{Ext}_{\mathcal{S}}^*(N, k) \longrightarrow 0$$

so that in particular

$$(13) \quad P_{\mathcal{S}}^{\overline{M}}(z) = P_{\mathcal{S}}^{\overline{M}/N}(z) + P_{\mathcal{S}}^N(z)$$

Now the exact sequence (11) shows that $P_{\mathcal{S}}^N(z) = 1 + 5zP_{\mathcal{S}}(z)$ and \overline{M}/N is generated by the 4 elements i, j, k, l which are annihilated by the maximal ideal of \mathcal{S} . Therefore $P_{\mathcal{S}}^{\overline{M}/N}(z) = 4P_{\mathcal{S}}(z)$ so that $P_{\mathcal{S}}^{\overline{M}}(z) = 1 + (4 + 5z)P_{\mathcal{S}}(z)$ and finally using (8)

$$(14) \quad P_{\overline{R}}(z) = \frac{P_{\mathcal{S}}(z)}{1 - z(1 + (4 + 5z)P_{\mathcal{S}}(z))}$$

Now, if we combine (14), rewritten in the form

$$\frac{1}{P_{\overline{R}}(z)} = \frac{1 - z}{P_{\mathcal{S}}(z)} - 4z - 5z^2$$

with (7) we obtain at last:

$$\frac{1}{P_{\overline{R}_{197}}(z)} = \frac{1 - z}{P_{\mathcal{S}}(z)} - 4z - 4z^2$$

whose more precise graded form is deduced as follows:

1) Replace the formula (7) by

$$(15) \quad \frac{1}{P_{\overline{R}_{197}}(x, y)} = \frac{1}{P_{\overline{R}}(x, y)} + x^2 y^4$$

2) Replace the formula (8) by

$$(16) \quad P_{\overline{R}}(x, y) = \frac{P_{\mathcal{S}}(x, y)}{1 - xyP_{\mathcal{S}}^{\overline{M}}(x, y)}$$

3) Replace the formula (13) by

$$(17) \quad P_{\mathcal{S}}^{\overline{M}}(x, y) = P_{\mathcal{S}}^{\overline{M}/N}(x, y) + P_{\mathcal{S}}^N(x, y)$$

where $P_{\mathcal{S}}^{\overline{M}/N}(x, y) = 4yP_{\mathcal{S}}(x, y)$ and $P_{\mathcal{S}}^N(x, y) = (4xy^2 + xy^3)P_{\mathcal{S}}(x, y)$ so that

$$P_{\mathcal{S}}^{\overline{M}}(x, y) = 4yP_{\mathcal{S}}(x, y) + 4(xy^2 + xy^3)P_{\mathcal{S}}(x, y)$$

and therefore finally

$$\frac{1}{P_{\overline{R}}(x, y)} = \frac{1 - xy}{P_{\mathcal{S}}(x, y)} - 4xy^2 - 4x^2y^3 - x^2y^4$$

so that using (15) we get the

THEOREM 1.– Let R_{197} be the numerical semigroup ring generated by

$$t^{36}, t^{48}, t^{50}, t^{52}, t^{56}, t^{60}, t^{66}, t^{67}, t^{107}, t^{121}, t^{129}, t^{135}$$

as a subring of $k[t]$ (k a field of characteristic 0) and let \overline{R}_{197} be R_{197} divided by the non-zero divisor t^{36} . Both these rings are Gorenstein rings and the bigraded Poincaré-Betti series for \overline{R}_{197} where the first 7 of the remaining 11 generators are given the degree 1 and the last 4 generators are given the degree 2 is given by the formula

$$\frac{1}{P_{\overline{R}_{197}}(x, y)} = \frac{1 - xy}{P_{\mathcal{S}}(x, y)} - 4xy^2 - 4x^2y^3$$

where \mathcal{S} is the ring given in (6) above and

$$1/P_{\mathcal{S}}(x, y) = (1 + 1/x)/\mathcal{S}^!(xy) - \mathcal{S}(-xy)/x$$

where $\mathcal{S}(t) = 1 + 6t + 10t^2 + t^3$ is the Hilbert series of \mathcal{S} and

$$\mathcal{S}^!(t) = \frac{1}{(1+t)(1-2t)^2(1-3t+t^2)} \prod_{n=2}^{\infty} \frac{(1+t^{2n-1})^2}{(1-t^{2n})^2}$$

is the Hilbert series of the Koszul dual of \mathcal{S} .

This last series will be determined in the next section.

Remark 1.– If we put $y = 1$ in all the formulae in Theorem 1, we get the ordinary Poincaré-Betti series of Betti numbers.

Remark 2.– It follows from what we have proved above that the natural map $\overline{R} \rightarrow \overline{R}/(h)$ is *large* in the sense of Levin [Le1], i.e. the natural map

$$(18) \quad \text{Ext}_{\overline{R}/(h)}^*(k, k) \rightarrow \text{Ext}_{\overline{R}}^*(k, k)$$

is a monomorphism.

(In his lecture notes about Golod homomorphisms [Le2], Levin calls such maps “co-Golod” maps.)

Indeed, the map $\overline{R} \rightarrow \overline{R}/(h)$ is the same as the natural map (here $N = (h)$ in \overline{M})

$$(19) \quad \mathcal{S} \propto \overline{M} \rightarrow \mathcal{S} \propto \overline{M}/N$$

Furthermore it is known that the ext-algebra of any trivial extension $C \propto L$ (where L is a C -module sits in the middle of a Hopf algebra extension:

$$k \rightarrow T(s^{-1}\text{Ext}_C^*(L, k)) \rightarrow \text{Ext}_{C \propto L}^*(k, k) \rightarrow \text{Ext}_C^*(k, k) \rightarrow k$$

where T is the graded tensor algebra and s^{-1} means that we push the degrees upwards one step. Thus if we take the extalgebras in (19) we obtain that (18) is a monomorphism if we can prove that

$$(20) \quad \text{Ext}_{\mathcal{S}}^*(\overline{M}/N, k) \rightarrow \text{Ext}_{\mathcal{S}}^*(\overline{M}, k)$$

is a monomorphism. But this is a consequence of (12) above.

2. Irrationality of the Poincaré-Betti series of \mathcal{S} .

Let \mathcal{S} be the ring

$$(21) \quad \mathcal{S} = \frac{\mathbf{Q}[b, c, d, e, f, g]}{(b^2, c^2 - bd, cd, d^2 - be, e^2 - df, de, bf, ef - cg, eg, f^2, g^2)}$$

We have already published in [Fro-Ro] and [Ro-St] the result that

$$(22) \quad 1/P_{\mathcal{S}}(z) = (1 + 1/z)/\mathcal{S}^!(z) - (1 - 6z + 10z^2 - z^3)/z$$

where

$$\mathcal{S}^!(z) = \frac{1}{(1+z)(1-2z)^2(1-3z+z^2)} \prod_{n=2}^{\infty} \frac{(1+z^{2n-1})^2}{(1-z^{2n})^2}$$

Here we will briefly indicate a general way of obtaining this result as a part of a general theory. First we observe that the hilbert series of \mathcal{S} is $1 + 6t + 10t^2 + t^3$. Furthermore the third power of the maximal ideal $m = (b, c, d, e, f, g)$ of \mathcal{S} is generated by bdg . Thus $m^3 = (b)m^2$ and since $b^2 = 0$ we can use another theorem of Levin, namely Theorem 2.12, page 33 of [Le2] for $n = 3$ which says that

$$\mathcal{S} \longrightarrow \mathcal{S}/(bdg)$$

is a Golod map and that

$$(23) \quad P_{\mathcal{S}/m^3}(z) = P_{\mathcal{S}}(z)/(1 - z^2 P_{\mathcal{S}}(z))$$

Therefore it is sufficient to concentrate our homological efforts on the ring $(T, n) = \mathcal{S}/m^3$ for which the cube of the maximal ideal n is 0. But now we can use the result of Clas Löfwall [Lö2] which says that

$$(24) \quad 1/P_T(z) = (1 + 1/z)/T^!(z) - T(-z)/z$$

where $T(-z) = 1 - 6z + 10z^2$ is the hilbert series of T at $-z$ and $T^!(z)$ is the hilbert series of the Koszul dual $T^!$ of T . It therefore remains to determine this last hilbert series. Recall that it follows from (21) and [Lö2] that

$$(25) \quad T^! = \frac{k < B, C, D, E, F, G >}{([B, C], C^2 + [B, D], D^2 + [B, E], [C, E], [C, F], E^2 + [D, F], [B, G], [E, F] + [C, G], [D, G], [F, G])}$$

where $k < B, C, D, E, F, G >$ is the free associative algebra in the variables B, C, D, E, F, G of degree 1 which are dual to b, c, d, e, f, g and where $[,]$ denotes the graded commutator, so that e.g. $[B, C] = BC + CB$ etc. Now we observe that $T^!$ is the enveloping algebra of the graded Lie algebra $\eta = \eta_T$ which is the quotient of the free graded Lie algebra on the generators B, C, D, E, F, G of degree 1 by the Lie ideal generated by the relations in (25). Thus if

$$\eta = \eta_1 \oplus \eta_2 \oplus \dots \oplus \eta_n \oplus \dots$$

the Poincaré-Birkhoff-Witt theorem tells us that

$$T^!(z) = \prod_{n \geq 1} \frac{(1 + z^{2n-1})^{\eta_{2n-1}}}{(1 - z^{2n})^{\eta_{2n}}}$$

We now turn to the problem of determining the η_i .

For this an essential role is played by the programme `liedim` written by Clas Löfwall [Lö1]. Recall that this programme (which runs under Mathematica and can be downloaded from <http://www2.math.su.se/~clas/liedim/>) works as follows: First start Mathematica. Then read in the input file `liedim.m` taken from [Lö1]. Then read in an input file which in the case (25) looks like (from now on we write b, c, d, e, f, g instead of B, C, D, E, F, G).

```

generators={b,c,d,e,f,g}
gensigns={1,1,1,1,1,1}
relations={lie[b, c], sq[c]+lie[b,d],sq[d]+lie[b,e],lie[c,e],lie[c,f],sq[e]+lie[d,f],
lie[b,g],lie[e,f]+lie[c,g],lie[d,g],lie[f,g]}

```

Now a command like *e.g*

```
maxdegree[7]
```

gives after a few seconds the result

```
{ 6, 11, 11, 18, 38, 79, 158}
```

Here the different numbers are the ranks of the η_i s for $i = 1, 2, 3, \dots, 7$. Thus there are 6 generators and in degree 2 there are 11 generating elements which in the programme are denoted by

```
modbas[2,1],modbas[2,2],...,modbas[2,11]
```

There is a command **def** which shows how the **modbas**-elements look in the Lie algebra η , so that *e.g.* **def[modbas[2,5]]**=**lie[e,d]** etc. There is the inverse of that command called **fed**, so that **fed[lie[e,d]]** = **modbas[2,5]**. Furthermore there is a command **ideal** which gives the ideal in the big liealgebra generated in a certain degree by given **modbas** elements. Thus, for example, **ideal[7,{modbas[3,5]}]** gives the graded vector space part in η_7 of the ideal generated by **modbas[3,5]**. These commands can be combined with ordinary Mathematica commands so that the combined command:

```
For[n=1,n<12,n++,Print[Length[ideal[7,{modbas[3,n]}]]][n]]
```

gives the result:

```

1[1]
53[2]
20[3]
15[4]
20[5]
15[6]
1[7]
52[8]
72[9]
52[10]
68[11]

```

which shows that the element **modbas[3,1]** generates an ideal which in degree 7 is one-dimensional, that **modbas[3,2]** generates an ideal which in degree 7 is of dimension 53 etc. In particular **modbas[3,1]** and **modbas[3,7]** in η_3 generate a unique very small ideal in η (it can be proved to be of dimension 2 in all degrees ≥ 3) which can be proved to be nilpotent (thus solvable). This small ideal will be denoted $rad(\eta)$ since it is a kind of radical of η and $\eta/rad(\eta)$ should be “semisimple” i.e. a product of “simple” lie algebras. This is indeed true if we restrict ourselves to “virtual” assertions, i.e. for results that are true in a high degree (in this case in degrees ≥ 3). The word “virtual” is inspired by Serre’s terminology in [Ser,section 1.8]. Let us be more precise: We have **def[modbas[3,1]]**=**lie[e, lie[b, b]]** and **def[modbas[3,7]]**=**lie[f, lie[f, d]]** so we now study the new liealgebra $\bar{\eta} = \eta / (\text{lie}[e, \text{lie}[b, b]], \text{lie}[f, \text{lie}[f, d]])$

Now for this lie algebra we have only 9 elements in degree 3, denoted by **modbas[3,i]** for $i = 1, \dots, 9$. Note that e.g. **modbas[3,2]** in this new lie algebra $\bar{\eta}$ is **lie[e, lie[e, b]]** whereas in the *old* lie algebra η **modbas[3,2]** is **lie[e, lie[d, c]]**. But a little experimentation shows that the 9 new elements of degree 3 in $\bar{\eta}$ can now be divided into three parts

$$J11 = \{\text{modbas}[3,3], \text{modbas}[3,5]\} \quad J12 = \{\text{modbas}[3,2], \text{modbas}[3,4]\}$$

and

$$J2 = \{\text{modbas}[3,9], \text{modbas}[3,8], \text{modbas}[3,7], \text{modbas}[3,6], \text{modbas}[3,1]\}$$

These parts are “orthogonal” to each other in the sense that the command `ann` in `liedim` gives that

$$\text{ann}[J2,3,3] = J11 \cup J12 \text{ and } \text{ann}[J11 \cup J12,3,3] = J2$$

and

$$\text{ann}[J11,3,3] = J12 \cup J2 \text{ and } \text{ann}[J12,3,3] = J11 \cup J2$$

Here the command `ann` is the annihilator command, written at my request by Clas Löfwall: Type `?ann` in `liedim` and the answer is:

`ann[a,t,s]` gives a basis for the elements in degree `s` which multiply the `modbas`-elements, which are of degree `t`, in the list `a` to zero

Note that multiplication in $\bar{\eta}$ of e.g. the elements `modbas[3,3]` and `modbas[3,5]` in `J11` and the all the elements in `J12` is given as follows

$$\text{op}[\text{def}[\text{modbas}[3,3]], J12] \text{ and } \text{op}[\text{def}[\text{modbas}[3,5]], J12]$$

and the result is $\{0, 0\}$ in both cases. One can continue higher up by induction using the Jacobi identity and find that `J11`, `J12` and `J2` are annihilating each other in all positive degrees. Next we calculate the lie algebra dimensions of the lie algebras $\bar{\eta}/J11$, $\bar{\eta}/J12$, and $\bar{\eta}/J2$. We get that the hilbert series of the enveloping algebra of the Lie algebra ideal generated by `J11` is $(1+z)(1-z)^3/(1-2z)$ (and the same for `J12`) and for `J2` we get $(1+z)^2(1-z)^5/(1-3z+z^2)$. It follows that the hilbert series of the enveloping algebra of $\bar{\eta}$ is

$$(26) \quad \frac{1}{(1+z)(1-2z)^2(1-3z+z^2)}$$

It remains to study the extension

$$(27) \quad 0 \longrightarrow \text{rad}(\eta) \longrightarrow \eta \longrightarrow \bar{\eta} \longrightarrow 0$$

Recall that $\text{rad}(\eta)$ is generated by `modbas[3,1]=lie[e, lie[b, b]]` and `modbas[3,7]=lie[f, lie[f, d]]` in η . In the following we will abbreviate expressions of this form as `ebb` and `ffd`.

The degree n part of $\text{rad}(\eta)$ is given by `ideal[n, {modbas[3,1], modbas[3,7]}]`. Thus we get

$$(28) \quad \begin{aligned} \text{ideal}[4, \{\text{modbas}[3,1], \text{modbas}[3,7]\}] &= \{\text{modbas}[4,4], \text{modbas}[4,12]\} = \{\text{eebb}, \text{ffeb}\} \\ \text{ideal}[5, \{\text{modbas}[3,1], \text{modbas}[3,7]\}] &= \{\text{modbas}[5,9], \text{modbas}[5,26]\} = \{\text{ebfbb}, \text{ffeeb}\} \\ \text{ideal}[6, \{\text{modbas}[3,1], \text{modbas}[3,7]\}] &= \{\text{modbas}[6,19], \text{modbas}[6,53]\} = \{\text{eebfbb}, \text{ffdeeb}\} \\ \text{ideal}[7, \{\text{modbas}[3,1], \text{modbas}[3,7]\}] &= \{\text{modbas}[7,21], \text{modbas}[7,100]\} = \{\text{ebfbfbb}, \text{ffedeeb}\} \\ \text{ideal}[8, \{\text{modbas}[3,1], \text{modbas}[3,7]\}] &= \{\text{modbas}[8,61], \text{modbas}[8,206]\} = \{\text{eebfbfbb}, \text{ffdedeeb}\} \end{aligned}$$

etc. All this indicates that

$$\text{rad}(\eta) = \bigoplus_{i \geq 3} kc_i \oplus kd_i$$

where the basis elements c_i and d_i are given in (28). Furthermore the multiplication in η is given by `op[def[modbas[s,t], modbas[u,v]]`. It follows that the ideal $\text{rad}(\eta)$ is abelian as far as we calculate and that the generators b, c, d, e, f, g operate very explitley e.g. as follows: `op[generators, modbas[7,21]] = {0, 0, 0, modbas[8, 61], 0, 0}` and `op[generators, modbas[7,100]] = {0, 0, modbas[8, 206], 0, 0, 0}` etc. and all this seems to subsist in all higher dimensions, and there is a beginning of a 2-cocycle $\varphi : \bar{\eta} \times \bar{\eta} \longrightarrow \text{rad}(\eta)$ that describes the extension:

$$0 \longrightarrow \text{rad}(\eta) \longrightarrow \eta \longrightarrow \bar{\eta} \longrightarrow 0$$

This is proved in the following way: first one observes one can prove by induction that the ideal generated by `ebb` and `ffd` is of dimension ≤ 2 in all positive degrees. Next one puts $L = \bigoplus_{i \geq 3} ke_i \oplus kf_i$ where the e_i and f_i are basis elements of degree i . Then one lets the variables b, c, d, e, f, g operate on these elements in

a way inspired by the above. Then one forms the extension $\bar{\eta} \times_{\varphi} L$. Now there is a natural *onto* map from η to $\bar{\eta} \times_{\varphi} L$ since the quadratic relations in η are mapped to 0-relations in $\bar{\eta} \times_{\varphi} L$. Thus $\text{rad}(\eta)$ which is of dimension ≤ 2 in all degrees is mapped onto L . Therefore it must be an isomorphism.

3. Final remarks

We have proved that the ring R_{197} is a Gorenstein numerical semigroup ring having an explicit transcendental Poincaré-Betti series, and we have said that the same result remains true for the rings R_n for $n = 199, 201, 203, \dots$. Let us just illustrate this by comparing R_{197} and R_{199} . In both cases the rings $R_{197}/(a)$ and $R_{199}/(a)$ (where $a = t^{36}$) are “the same” but the gradings of the remaining generators $b, c, d, e, f, g, h, i, j, k, l$ are different: Indeed the gradings are $(48, 50, 52, 56, 60, 66, 67, 107, 121, 129, 135)$ for $R_{197}/(a)$ and $(48, 50, 52, 56, 60, 66, 69, 109, 123, 131, 137)$ for $R_{199}/(a)$. But we see in (5) that if we choose the gradings $c_1 = 48, c_2 = 52$ we obtain that the gradings of $c = 50, d = 52, e = 56, f = 60, g = 66, h = c_3, i = c_3 + 40, j = c_3 + 54, k = c_3 + 86, l = c_3 + 68$. Thus if $h = c_3 = 67$, then $(i, j, k, l) = (107, 121, 129, 135)$ i.e. the case $R_{197}/(a)$ which we have studied above. But if $h = c_3 = 69$, then $(i, j, k, l) = (109, 123, 131, 137)$, i.e. we are in case $R_{199}/(a)$. This shows together with (5) that the cases $R_{197}/(a)$ and $R_{199}/(a)$ can be regraded with $(b, c, d, e, f, g, h, i, j, k, l) = (1, 1, 1, 1, 1, 1, 2, 2, 2, 2)$ so that they become isomorphic. However the graded rings R_{197} and R_{199} can *not* be regraded so that they become isomorphic. Example: If we look for the possible gradings of the ideal J of (3) we should as earlier solve the more complicated equations, where as above the relation $b^2 - af$ is interpreted as $2b - a - f = 0$ etc.:

$$\begin{array}{llll}
& & & a + d + i - f - l = 0 \\
2b - a - f = 0 & c + i - a - j = 0 & d + f + h - c - k = 0 & g + k - f - l = 0 \\
c + c - b - d = 0 & a + e + h - d - i = 0 & a + a + i - c - k = 0 & a + b + d + f - h - k = 0 \\
c + d - a - g = 0 & a + f + h - e - i = 0 & b + g + h - d - k = 0 & a + a + k - g - l = 0 \\
d + d - b - e = 0 & b + c + h - a - k = 0 & f + j - d - k = 0 & a + b + d + g - h - l = 0 \\
a + a + a - b - f = 0 & b + d + h - f - i = 0 & c + g + h - b - l = 0 & a + d + f + g - i - i = 0 \\
d + e - b - f = 0 & a + g + h - b - j = 0 & d + g + h - c - l = 0 & a + f + g + g - i - j = 0 \\
e + e - d - f = 0 & b + e + h - a - l = 0 & e + k - c - l = 0 & b + h + j - i - k = 0 \\
e + f - c - g = 0 & c + j - a - l = 0 & f + f + h - d - l = 0 & b + f + h + h - i - l = 0 \\
a + a + b - f - f = 0 & c + e + h - d - j = 0 & g + j - d - l = 0 & j + j - i - l = 0 \\
a + a + c - e - g = 0 & g + i - d - j = 0 & e + g + g - h - j = 0 & j + k - b - h - l = 0 \\
a + a + f - g - g = 0 & b + f + g - h - i = 0 & e + g + h - f - k = 0 & f + h + k - j - l = 0 \\
a + b + c - h - h = 0 & c + f + h - b - k = 0 & a + b + i - e - l = 0 & k + k - e - h - l = 0 \\
a + d + h - b - i = 0 & e + j - b - k = 0 & f + g + h - a - a - j = 0 & a + i + j - k - l = 0 \\
& & & e + i + i - l - l = 0
\end{array}$$

The solutions of these 54 equations in 12 unknowns are:

$$a = 36c_1/67, b = 48c_1/67, c = 50c_1/67, d = 52c_1/67, e = 56c_1/67, f = 60c_1/67,$$

$$g = 66c_1/67, h = c_1, i = 107c_1/67, j = 121c_1/67, k = 129c_1/67, l = 135c_1/67$$

where c_1 is a constant and since 67 is a prime number, the minimal possible integral choice is $c_1 = 67$, and so we find the unique grading of R_{197} . But if we do the same reasoning for R_{199} we still get 54 relations between the $a, b, c, d, e, f, g, h, i, j, k$ corresponding to, but different from (3):

$$\begin{aligned}
& b^2 - af, c^2 - bd, cd - ag, d^2 - be, de - bf, a^3 - bf, e^2 - df, ef - cg, a^2b - f^2, a^2c - eg, a^2f - g^2, a^2g - h^2, \\
& adh - bi, ci - aj, aeh - di, afh - ei, bch - ak, bdh - fi, agh - bj, cj - al, beh - al, gi - dj, ceh - dj, dfg - hi, \\
& ej - bk, cfh - bk, a^2i - ck, dfh - ck, fj - dk, bgh - dk, cgh - bl, ek - cl, dgh - cl, gj - dl, f^2h - dl, egh - fk,
\end{aligned}$$

$$\begin{aligned}
& f g^2 - h j, a b i - e l, f g h - a^2 j, g k - f l, a d i - f l, a b c g - h k, a^2 k - g l, a b e g - h l, a c g^2 - i^2, b d g^2 - i j, b h j - i k, j^2 - i l, \\
(29) \quad & b f h^2 - i l, j k - b h l, f h k - j l, k^2 - e h l, a i j - k l, e i^2 - l^2
\end{aligned}$$

and, when we want to analyze the possible gradings of R_{199} we should now solve the 54 linear equations for the $a, b, c, d, e, f, g, h, i, j, k$ corresponding to (29). The solutions to these equations are now:

$$\begin{aligned}
& a = 12c_1/23, b = 16c_1/23, c = 50c_1/69, d = 52c_1/69, e = 56c_1/69, f = 20c_1/23, \\
(30) \quad & g = 22c_1/23, h = c_1, i = 109c_1/69, j = 41c_1/23, k = 131c_1/69, l = 137c_1/69
\end{aligned}$$

But here 23 is a prime number and the minimum possible choice of c_1 so that all the gradings in (30) become integers are $c_1 = 3 \cdot 23 = 69$ and this gives again the original grading.

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